a) Boundary Conditions (along the edges, x = const):

$$\sigma_x = \phi_{,yy} = \overline{P}_x$$

$$\tau_{xy} = -\phi_{,xy} = \overline{P}_y \qquad (5)$$

b) Boundary Compatibility Conditions:

$$\epsilon_y = 1/E\,(\phi_{,xx} - \nu\phi_{,yy}) = 0$$

$$\epsilon_{y,x} - \gamma_{xy,y} = \{\phi_{,xx} - 2(1+\nu)\,\phi_{,yy}\}_{,x} = 0 \qquad (6)$$

Equation (6) may also be rewritten in terms of displacements as

$$\epsilon_y = v_{,y} = 0$$

$$\epsilon_{y,x} - \gamma_{xy,y} = u_{,yy} = 0 \qquad (7)$$

which represent a constant value of v and a linear variation of u (constant rotation). Since consideration $u = v = 0$ on this boundary satisfies Eq. (7), the stress field obtained as a solution of Eq. (4) with boundary conditions Eqs. (5) and (6) corresponds to the solution of the problem with homogeneous boundary conditions on S_u. The boundary conditions on the edge y = constant have similar form.

Thus, the stresses in this class of mixed boundary-value problems in which displacement boundary conditions are homogeneous can be completely formulated in terms of stresses as solution to Eqs. (4-6). This process eliminates the consideration of displacement boundary conditions on S_u.

Example

As an example we consider the bending of a beam fixed at both ends ($x = 0$ and a). It is subjected to a uniformly distributed load of intensity q. The potential function for this example can be expressed as follows:

$$\delta\Pi_s = \delta\Pi_s^{(1)} + \delta\Pi_s^{(2)}$$

where

$$\delta\Pi_s^{(1)} = \int_o^a M\delta k\,dx - \int_o^a q\,\delta\omega\,dx + \delta(\lambda,\omega)\Big|_o^a + \delta(\lambda_2\omega_{,x})\Big|_o^a \qquad (8)$$

$$\delta\Pi_s^{(2)} = \int_o^a k\,\delta M\,dx + \delta\int_o^a \lambda(M_{,xx} - q)\,dx \qquad (9)$$

where M is the bending moment and k is the curvature. In what follows we consider $\Pi_s^{(2)}$. M and k may be expressed in terms of a stress function ϕ as

$$M = \phi_{,xx}$$

$$k = -\phi_{,xx}/EI \qquad (10)$$

where E is Young's Modulus and I is the second moment of the area. The stress function ϕ, required to satisfy the equilibrium equation

$$M_{,xx} = q \quad \text{or} \quad \phi_{,xxxx} = q \qquad (11)$$

is introduced in the potential function through the Lagrangian multiplier λ.

Using Eqs. (10) and (11), the expression for $\delta\Pi_s^{(2)}$ is rewritten as

$$\delta\Pi_s^{(2)} = -\int_o^a (\phi_{,xx}/EI)\,\delta\phi_{,xx}\,dx + \delta\int_o^a \lambda(\phi_{,xxxx} - q)\,dx$$

Using the standard variational procedure, the governing differential equations and boundary conditions are obtained as

$$\phi_{,xxxx} = q \qquad (12)$$

$$[(\phi/EI) + \lambda]_{,xxxx} = 0 \qquad (13)$$

with boundary conditions

$$\phi = 0;\ \phi_{,x} = 0 \text{ at } x = 0 \text{ and } a \qquad (14)$$

$$\lambda = 0;\ \lambda_{,x} = 0 \text{ at } x = 0 \text{ and } a \qquad (15)$$

Equations (12-15) represent the IFM. It may be readily verified that the solution of Eq. (12) along with the boundary conditions of Eq. (14) completely defines ϕ and hence the moment distribution in this beam. Thus, the moment response of the clamped beam with displacement boundary conditions can be obtained using the integrated force method without any reference to displacements on the boundary or in the field. Furthermore, the solution of Eq. (13) along with the boundary conditions of Eq. (15) yields the Lagrangian multiplier, which can be readily identified as the normal displacement distribution in the beam.

The present study is an attempt to establish the feasibility of obtaining complete information on stresses in a mixed boundary-value problem with homogeneous displacement boundary conditions. In principle, this can be extended to include body forces and elastic supports, etc., by appropriately including them in either part of the potential function given in Eq. (1). In such a case, the decoupling of the two problems cannot always be expected. Further study in this direction will be interesting and useful.

References

[1]Patnaik, S. N., "An Integrated Force Method for Discrete Analysis," *International Journal for Numerical Methods in Engineering*, Vol. 6, No. 2, 1973, pp. 237-251.

[2]Patnaik, S. N. and Joseph, K. T., "Compatibility Conditions from Deformation Displacement Relations," *AIAA Journal*, Vol. 23, Aug. 1985, pp. 1291-1293.

[3]Patnaik, S. N. and Joseph, K. T., "Generation of the Compatibility Matrix in the Integrated Force Method," *Computer Methods in Applied Mechanics and Engineering*, Vol. 55, May 1986, pp. 239-257.

[4]Patnaik, S. N., "Integrated Force Method Versus the Standard Force Method," *International Journal of Computers and Structures*, Vol. 22, No. 2, 1986, pp. 151-163.

[5]Patnaik, S. N. and Yadagiri, S., "Frequency Analysis of Structures by the Integrated Force Method," *Journal of Sound and Vibration*, Vol. 83, July 1982, pp. 93-109.

[6]Patnaik, S. N. and Gallagher, R. H., "Gradients of Behaviour Constraints and Reanalysis via the Integrated Force Method," *International Journal of Numerical Methods in Engineering*, Vol. 23, No. 12, 1986, pp. 2205-2212.

[7]Patnaik, S. N., "The Variational Energy Formulation for the Integrated Force Method," *AIAA Journal*, Vol. 24, Jan. 1986, pp. 129-137.

[8]Patnaik, S. N. and Nagaraj, M. S., "Analysis of Continuum by the Integrated Force Method," *Computers and Structures*, Vol. 26, No. 6, 1987, pp. 899-906.

On a General Method of Vibration Analysis in Curvilinear Coordinates

Sunil K. Sinha*

General Electric Company, Louisville, Kentucky

MOST of the dynamic problems in elasticity in curvilinear coordinates are formulated to solve a particular type of

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*Senior Analytical Engineer.

problem only and no general method that can be used for any coordinate system is readily available. For such specialized applications, the mathematical derivation is usually limited to either cylindrical or spherical coordinates only. In this Note, the most general form of stiffness and mass matrices to be used in free-vibration equations are derived from Hamilton's principle. These equations are presented for isotropic, linear-elastic material and are valid in any coordinate system (including nonorthogonal).

For convenience we will use standard tensor notations[1] where repeated indices refer to additions and superscripts and subscripts stand for contravariant and covariant components of tensor, respectively. Similarly, g stands for the metric tensor, and a comma refers to the covariant derivative.

If $\boldsymbol{R}(x_i)$ is the position vector of any point on an elastic body in the curvilinear space, then the base vector for this system can be written as

$$g_i = \frac{\partial \boldsymbol{R}(x_i)}{\partial x_i} \tag{1}$$

We also have the covariant metric tensor

$$g_{ij} = g_i \cdot g_j \tag{2}$$

and contravariant metric tensor

$$g^{ij} = g^i \cdot g^j \tag{3}$$

The covariant and contravariant components of the metric tensor are related by "Kronecker delta"

$$\delta_j^i = g^{ik} g_{kj} \tag{4}$$

We introduce the following two additional notations, commonly known as Christoffel symbols of the first and second kinds, respectively.

First kind:

$$[ik,q] = \frac{1}{2}\left\{\frac{\partial}{\partial x_i}(g_{kq}) + \frac{\partial}{\partial x_k}(g_{iq}) - \frac{\partial}{\partial x_q}(g_{ik})\right\} \tag{5}$$

Second kind:

$$\Gamma_{ik}^p = g^{pq}[ik,q] = g^{p1}[ik,1] + g^{p2}[ik,2] + g^{p3}[ik,3] \tag{6}$$

The covariant derivative of a second-order mixed tensor T_i^j is defined as

$$[T_i^j]_{,k} = \frac{\partial T_i^j}{\partial x_k} - \Gamma_{ik}^p T_p^j + \Gamma_{qk}^j T_i^q \tag{7}$$

Let us consider the deformed configuration of the body under a system of external surface traction p_i, such that a point on the body originally at $\boldsymbol{R}(x_i)$ deforms by an amount $\boldsymbol{u}(x_i)$. Using u_i for the displacement vector in the tensor notation and other standard nomenclatures from linear elasticity,[2] we obtain the following dynamical relationships:

Strain-displacement relation:

$$\epsilon_{ij} = ½\,(u_{i,j} + u_{j,i}) \tag{8}$$

Stress-strain relation:

$$\sigma_{ij} = 2\mu\,\epsilon_{ij} + \lambda\, g^{kr}\,\epsilon_{kr}\, g_{ij} \tag{9}$$

Work due to external forces:

$$\Omega = \int_{\text{area}} g^{ij} P_i u_j \,\mathrm{d}A \tag{10}$$

Strain energy:

$$U = \frac{1}{2}\int_{\text{vol}} g^{ip} g^{jq} \sigma_{ij} \epsilon_{pq}\,\mathrm{d}v \tag{11}$$

Potential energy:

$$V = U + \Omega \tag{12}$$

Kinetic energy:

$$T = \frac{1}{2}\int_{\text{vol}} \rho\, g^{ij} \dot{u}_i \dot{u}_j \,\mathrm{d}v \tag{13}$$

Lagrangian function:

$$L = T - V \tag{14}$$

Action integral:

$$\mathcal{A} = \int_{t_1}^{t_2} L\,\mathrm{d}t \tag{15}$$

where $\mathrm{d}v = \sqrt{G}\ \mathrm{d}x_1\ \mathrm{d}x_2\ \mathrm{d}x_3$, and $G = \det|g_{ij}|$. In free vibration of an elastic body, there are no external forces, that is, $P_i = 0$, resulting into $\Omega = 0$. Thus,

$$\mathcal{A} = \int_{t_1}^{t_2} (T - U)\,\mathrm{d}t \tag{16}$$

From the Hamilton's principle for a conservative system,[3] we have $\delta\mathcal{A} = 0$, which yields

$$\int_{t_1}^{t_2} \delta T\,\mathrm{d}t = \int_{t_1}^{t_2} \delta U\,\mathrm{d}t \tag{17}$$

Now, from Eq. (13) we have

$$\int_{t_1}^{t_2} \delta T\,\mathrm{d}t = \int_{\text{vol}} \rho\, g^{ij}\left[\int_{t_1}^{t_2} \dot{u}_j\,\delta\dot{u}_i\,\mathrm{d}t\right]\mathrm{d}v \tag{18}$$

On integrating the right-hand side of the above equation by parts, we obtain

$$\int_{t_1}^{t_2} \delta T\,\mathrm{d}t = \int_{\text{vol}} \rho\, g^{ij}\left[\dot{u}_j\,\delta u_i\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \delta u_i\,\ddot{u}_j\,\mathrm{d}t\right]\mathrm{d}v$$

Since, $\dot{u}_j\,\delta u_i\,\big|_{t_1}^{t_2} = 0$, the above equation further simplifies into

$$\int_{t_1}^{t_2} \delta T\,\mathrm{d}t = -\int_{t_1}^{t_2}\int_{\text{vol}} \rho\, g^{ij}\,\ddot{u}_j\,\mathrm{d}v\,(\delta u_i)\,\mathrm{d}t \tag{19}$$

Similarly, from Eq. (11) we have

$$\int_{t_1}^{t_2} \delta U\,\mathrm{d}t = \int_{t_1}^{t_2}\int_{\text{vol}} (2\mu\, g^{ip} g^{jq}\,\epsilon_{pq} + \lambda\, g^{kr} g^{ij}\,\epsilon_{kr})\,\delta\epsilon_{ij}\,\mathrm{d}v\,\mathrm{d}t \tag{20}$$

On substituting the strain-displacement relation from Eq. (8), Eq. (20) reduces to

$$\int_{t_1}^{t_2} \delta U\,\mathrm{d}t = \int_{t_1}^{t_2}\int_{\text{vol}} (2\mu\, g^{ip} g^{jq}\,u_{p,q} + \lambda\, g^{kr} g^{ij}\,u_{k,r})\,\delta u_{i,j}\,\mathrm{d}v\,\mathrm{d}t \tag{21}$$

At this point, we can substitute Eqs. (19) and (21) in Eq. (17). Since the resulting expression would be true for any value of time, we can drop the integration over t, which yields

$$\int_{\text{vol}} \rho g^{ij} \ddot{u}_j \, dv \, (\delta u_i) + \int_{\text{vol}} (2\mu \, g^{ip} g^{jq} u_{p,q} + \lambda \, g^{kr} g^{ij} u_{k,r}) \, dv \, (\delta u_{i,j}) = 0 \tag{22}$$

In a dynamic problem, u_i being the component of the displacement vector is a function of x_1, x_2, x_3, and t. Hence, by separating the variables in the spatial dimensions and the time domain, we can write

$$u_i(x_\ell, t) = N_i^f(x_\ell) W_f(t) \tag{23}$$

where $N_i^f(x_\ell)$ is the second-order mixed tensor and as such from Eq. (6),

$$[N_i^f],_j = \frac{\partial N_i^f}{\partial x_j} + \Gamma_{pj}^f N_i^p - \Gamma_{ij}^q N_q^f \tag{24}$$

We also have

$$\delta u_i = [N_i^f] \, \delta W_f \tag{25}$$

$$\delta u_{i,j} = [N_i^f],_j \, \delta W_f \tag{26}$$

and

$$\ddot{u}_j = [N_j^f] \, \ddot{W}_f \tag{27}$$

The substitution of these values in Eq. (22) yields

$$\left[\int_{\text{vol}} \rho \, g^{ij} N_j^f N_i^h \, dv\right] \ddot{W}_f \, \delta W_h + \left[\int_{\text{vol}} \{2\mu \, g^{ip} g^{jq} [N_p^f],_q [N_i^h],_j + \lambda \, g^{kr} g^{ij} [N_k^f],_r [N_i^h],_j\} \, dv\right] W_f \, \delta W_h = 0 \tag{28}$$

which is also the equation of free vibration for the elastic body and, in the usual matrix representation, is written as

$$[M] \, \ddot{W}_f + [K] \, W_f = 0 \tag{29}$$

In the finite-element method,[4] if $[N_i^f]$ is considered as a suitable displacement shape function in the curvilinear coordinate system, then the relationships for the mass matrix $[M]$ and the stiffness matrix $[K]$ quickly follow. Thus, for an assumed shape function $[N_i^f]$ and known contravariant metric tensor g^{ij}, the mass and stiffness matrices can be determined as follows:

$$[M] = \int_{\text{vol}} \rho \, [N_j^f] \, [g^{ij}] \, [N_i^h] \, dv \tag{30}$$

$$[K] = \int_{\text{vol}} \{2\mu \, [N_p^f],_q \, [g^{jq}] \, [g^{ip}] \, [N_i^h],_j + \lambda \, [N_k^f],_r \, [g^{kr}] \, [g^{ij}] \, [N_i^h],_j \} \, dv \tag{31}$$

The above expressions for the mass and stiffness matrices are the most general in nature and are good in any coordinate system.

References

[1]McConnell, A. J., *Application of Tensor Analysis,* Dover, New York, 1957, pp. 140–147.

[2]Sokolnikoff, I. S., *Mathematical Theory of Elasticity,* 2nd ed., McGraw-Hill, New York, 1957, p. 181.

[3]Langhaar, H. L., *Energy Methods in Applied Mechanics,* Wiley, New York, 1962, p. 234.

[4]Zienkowicz, O. C., *The Finite-Element Method,* McGraw-Hill, London, 1977, pp. 527–565.